

# Scaling Behavior in the Stable Marriage Problem

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## Abstract

We study the optimization of the stable marriage problem. All individuals attempt to optimize their own satisfaction, subject to mutually conflicting constraints. We find that the stable solutions are generally not the globally best solution, but reasonably close to it. All the stable solutions form a special sub-set of the meta-stable states, obeying interesting scaling laws. Both numerical and analytical tools are used to derive our results.

## I. INTRODUCTION

Optimization problems have become an interesting area of research in statistical physics. They usually require to find the minimum (or minima) of a global quantity (e.g. Hamiltonian). Spin-glasses are just one of the current well-known examples. In the context of social science or economy, decision-makers are individuals or companies. They have their own rather selfish goals to optimize, which are often conflicting. This situation, which is typical in game theory, cannot be described by a global Hamiltonian [1,2]. One of its simplest realizations is in the classical stable marriage problem [3–5]. Despite its simple definition the solutions have a very rich structure.

The stable marriage optimization problem does not require the "energy" to be the smallest possible, but that the resulting state be stable against the egoistic attempts of individuals to lower their own "energy". This new concept of equilibrium is typical in game theory where one deals with a number  $N$  of distinct agents, each of whom is trying to maximize his utility at the same time. It is not true in general that the state with the largest total utility will be an equilibrium state. Indeed it is possible that in such a state an agent would benefit from making an action which increases his utility at the expense of others. This leads to the concept of Nash equilibrium [6], which is a state characterized by the stability with respect to the action of any agent. In other words, a state is stable if any change in an agent's strategy is unfavorable for himself.

The marriage problem describes a system where two sets of  $N$  persons have to be matched pairwise. We shall assume that these sets are composed of men and women who are to be married. Clearly the marriage problem is applicable in many different contexts where two distinct sets have to be matched with the best satisfaction. For instance, one can consider  $N$  applicants facing  $N'$  jobs. Each applicant has a preference list of jobs and each job-owner ranks the applicants in order of preference. For the convenience of presentation we exclusively use the paradigm of marriage between men and women. Suppose the men and the women in the two sets know each other well. Based on his knowledge each man establishes

a wish-list of his desired women, in the descending priority order, i.e. on the top of his list is his dream girl; the bottom is a mate whom he has to marry only in the worst case when all the other women reject him. The women do exactly the same to the men. Note that in the convention of this model, everybody must marry.

In our model, we make the further simplifying assumption that each person's satisfaction depends on the rank of the partner he/she gets to marry. Thus the rank can be seen as a cost function. If the top choice is attained, the cost is the least, the bottom choice has the highest cost. Two men may happen to put the same woman as their top choice, or two women may happen to prefer the same man. There are necessarily conflicting wishes which give a special complexity to the problem. We shall deal here with the case in which each person's wish-list is randomly and independently established.

## II. THE MODEL

In order to set up the notation let us look at the following example of three men and three women. The preference lists for all persons are shown below.

1: 1 2 3	1: 3 1 2
2: 1 3 2	2: 1 2 3
3: 2 3 1	3: 3 2 1

Men's preference lists	Women's preference lists
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The detailed cost for each person can be set observing that, e.g. if man 1 marries woman 2 his cost is 2 since she is in the second position of his preference list and, vice versa, the cost for woman 2 is 1. If man 2 marries woman 1, his cost is 1 and if man 3 marries woman 3 his cost is 2. The costs of women 1 and 3 in these cases are 3 and 1, respectively.

It is convenient to introduce a representation of the lists in terms of rankings: We define the matrices  $F$  and  $H$  for women and men respectively, such that  $f(w, m)$  denotes the

position of a man  $m$  in the list of a woman  $w$ . Equivalently  $h(m, w)$  yields the rank of a woman  $w$  in  $m$ 's list. Rank plays here a similar role to energy in statistical mechanics, so we shall frequently refer to  $f(w, m)$  or  $h(m, w)$  as energies. A realization of the preference lists is also called an instance. A matching is a set of  $N$  pairs  $\mathcal{M} = \{(m_i, w_i), i = 1, \dots, N\}$ , and there are  $N!$  possible matchings in an instance of size  $N$ .

The problem is to find a *stable* matching  $\mathcal{M} = \{(m_i, w_i)\}$  such that one cannot find a man  $m_i$  and a woman  $w_j$  who are not married ( $i \neq j$ ) but would *both* prefer to marry each other rather than staying with their respective partners  $w_i$  and  $m_j$ . Such a couple is called a *blocking pair*. A blocking pair  $(m_i, w_j)$  is then such that  $f(w_j, m_i) < f(w_j, m_j)$  and  $h(m_i, w_j) < h(m_i, w_i)$ . If no such pair exists, the matching is called stable.

One can calculate the energy per person, for women and men in a given matching as

$$\epsilon_F(\mathcal{M}) = \frac{1}{N} \sum_{i=1}^N f(w_i, m_i), \quad \epsilon_H(\mathcal{M}) = \frac{1}{N} \sum_{i=1}^N h(m_i, w_i)$$

and the energy  $\epsilon(\mathcal{M}) = \epsilon_F(\mathcal{M}) + \epsilon_H(\mathcal{M})$  per couple. Here and in the following the subscripts  $F$  and  $H$  stand for women and men, respectively.

### III. THE GALE-SHAPLEY ALGORITHM

A lot of work has been spent in developing fast computer algorithms to find all stable matchings for a given instance of size  $N$  [4,5]. These algorithms are based on the classical Gale-Shapley (GS) algorithm [3] which assigns the role of proposers to the elements of one set, the men say, and of judges to the elements of the other.

The *man-oriented* GS algorithm starts from a man  $m$  making a proposal to the first woman  $w$  on his list. If she accepts they get married, if she refuses  $m$  goes on proposing to the next woman on his list.  $w$  accepts a proposal when either she is not engaged or she is engaged with a man  $m'$  worse than the one proposing ( $m$ ). In the latter case,  $m'$  will have to go on proposing to the woman following  $w$  on his list. When all men have run through their lists proposing until all women are married, the algorithm stops and the matching thus reached is a stable matching.

As an illustration let us consider the example of the preceding paragraph. The man-oriented GS algorithm goes as follows: man 1 proposes to woman 1 who accepts and they form the pair  $(1, 1)$ . Then 2 proposes to 1, but she refuses. So man 2 proposes to woman 3 and they get married. Finally man 3 happily marries woman 2. This results in the matching  $\mathcal{M}_H = \{(1, 1), (2, 3), (3, 2)\}$ .

The GS algorithm can be run reversing the roles (woman-oriented) to yield the woman-optimal stable matching. In our example, this leads to  $\mathcal{M}_F = \{(1, 1), (2, 2), (3, 3)\}$ .

The energies for men and women in these matchings are  $(\epsilon_H, \epsilon_F) = (4/3, 7/3)$  and  $(6/3, 5/3)$  respectively for  $\mathcal{M}_H$  and  $\mathcal{M}_F$ . As seen in this example who proposes is always better off than who judges.

It can be shown [4] that the man-oriented GS algorithm yields the man-optimal stable matching in the sense that no man can have a better partner in any other stable matching. It is rather surprising that though men get rejections nearly all the time and just one positive answer, they are far better off than women. On the other hand women who take the pleasure by saying no to almost all the suitors except one who is best among her suitors, will end up in a marriage that is the worst among all possible stable ones. The lesson is that the person who takes initiative is rewarded.

In order to quantify more precisely this statement, it is enough to observe that *i)* once a woman is first engaged, she will remain engaged (eventually with different men) forever, and that *ii)* the total energy for men in the man-optimal GS equals the total number of proposals men need to make to marry all women. Proposals by men define an intrinsic time in the algorithm. Imagine that at time  $t_k$  (i.e. after  $t_k$  proposals) the  $k^{\text{th}}$  woman gets engaged. In view of the randomness of the preference lists, the probability that the next proposal is addressed to one of the  $N - k$  free women is  $1 - k/N$ . On average, men will need  $N/k = \langle t_{k+1} - t_k \rangle$  proposals to engage one more woman. Since the total energy for men is  $N\epsilon_H = t_N$ , we find

$$\epsilon_H(\mathcal{M}_H) = \sum_{k=1}^N \frac{1}{k} \approx \log N + C \quad (1)$$

where  $C = 0.5772\dots$  is Euler's constant. Taking into account that men do not propose twice to the same woman yields a correction of  $\mathcal{O}((\log N)^2/N)$  to eq. (1). On the average each woman receives  $\log N + C$  proposals of men who are randomly distributed between 1 and  $N$  on her preference list. Keeping only the best proposal the women arrive at an energy of the order of  $\epsilon_F(\mathcal{M}_H) \simeq N/\log N$  which is much larger than the corresponding result (1) for the men.

#### IV. STABLE MATCHINGS IN THE LARGE $N$ LIMIT

Coming back to our  $N = 3$  example, it is also interesting to note that none of the two stable states we found is the one with the smallest energy, the “ground state”. Indeed the state  $\mathcal{M}_0 = \{(1, 2), (2, 1), (3, 3)\}$  has  $(\epsilon_H, \epsilon_F) = (5/3, 5/3)$ , and a total energy  $\epsilon(\mathcal{M}_0) = 10/3$  which is lower than those of the other two states. This state is however unstable. Indeed there is one blocking pair  $(1, 1)$ .

This simple example already shows some interesting features of the stable marriage problem: 1) minimum energy i.e. maximum global satisfaction does not imply stability (and vice versa) and 2) there can be more than one stable matching (the GS algorithm guarantees that there is always at least one stable matching).

In order to decrease the total energy of a given stable matching, it would be necessary that some individuals pay the price of accepting a worse partner than the one with whom they are actually married. But in absence of a supervising body (like government or parents) they do not consider to help others by switching. The inherent selfishness and conflicting optimizations in the stable marriage problem lead to stable solutions that are not globally optimal.

It is known that the average number of stable states in an instance of size  $N$  is proportional to  $N \log N$  [5]. Starting from the man-optimal Gale-Shapley solution all other stable matchings can be obtained by performing cyclic exchange processes

$$(\mu_1, \omega_1), \dots, (\mu_r, \omega_r) \rightarrow (\mu_1, \omega_2), \dots, (\mu_r, \omega_1) \quad (2)$$

within a properly chosen group of pairs of men  $\mu_i$  and women  $\omega_i$ . These exchange processes are called rotations. The number  $r$  of pairs involved in a rotation can be regarded as the "distance" between the two stable matchings connected by this rotation. Efficient algorithms exist for finding all rotations and therefore all stable matchings in a given instance [5]. In this way it has been found that stable states are organized in very peculiar graph-theoretical structures [5].

We shall first analyze the statistics of stable states and then return to the concept of rotations.

It is possible to derive a general relation between the energies of men and women in the stable states. Our aim is to evaluate the energy of men, given that of women in a stable state. We can then assume that a stable matching, in which woman  $w$  have energy  $\epsilon_w$ , exists. In order to find the stable matching we consider the "hypothetical" situation in which, for some reason, the women know that they can reach a stable matching where woman  $w$  has energy  $\epsilon_w$ . Knowing this, the best strategy of women, becomes that of refusing all propositions from men ranking higher than  $\epsilon_w$ . The best strategy for men, on the other hand, remains that of the GS algorithm. The dynamics of this modified GS algorithm will clearly reach the stable state foreseen by women: Each man  $m$  makes his proposals sequentially until he hits a woman  $w$  such that  $f(w, m) \leq \epsilon_w$ . In order to compute the average man energy, consider a man  $m$  and, in order to simplify the notations, let his wish list be  $h(m, w) = w$ , for  $w = 1, \dots, N^1$ . His proposal to the  $w^{\text{th}}$  woman, will fall randomly within 1 and  $N$  in the rankings of woman  $w$  and, according to the above strategy, it will be accepted with a probability  $\epsilon_w/N$ . Otherwise, if it is refused, man  $m$  will consider the  $w + 1^{\text{st}}$  woman. The probability that this man will get his  $q^{\text{th}}$  choice is then

$$P_H(q) = \prod_{w=1}^{q-1} \left(1 - \frac{\epsilon_w}{N}\right) \frac{\epsilon_q}{N}. \quad (3)$$

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<sup>1</sup>This can always be achieved with a permutation of women indices. Clearly  $h(m', w) \neq w$  for  $m' \neq m$ , in general.



We can now take the average on realizations of the above equation. Under the assumption that  $\epsilon_w$  are independent random variables with average  $\epsilon_F$ , we find

$$P_H(q) = \left(1 - \frac{\epsilon_F}{N}\right)^{q-1} \frac{\epsilon_F}{N}. \quad (4)$$

From this we can compute the average energy for the  $m^{\text{th}}$  man  $\epsilon_H = \sum q P_H(q)$ . This leads to the relation

$$\epsilon_H \epsilon_F = N. \quad (5)$$

In other words, *in stable matchings, for random instances of the marriage problem, the energies for men and women are inversely proportional.*

Of course, reversing the roles of men and women, one finds the same conclusion eq. (5) and a distribution  $P_F(q) \simeq \exp(-q/\epsilon_H)/\epsilon_H$  of women energies which depends on  $\epsilon_H$ .

It is now possible, returning to the assumption of independence of the  $\epsilon_w$ 's, to show that only a weak correlation exists so that equations (4) and (5) are exact in the limit  $N \rightarrow \infty$ . In order to do this we can run the above argument in its women-oriented version (with women proposing) to derive the joint probability distribution of the energies of two women. With independent men energies, it is clear that unless two women propose to the same man, there will be no correlation between their energies. The probability that both propose to the same man is of the order of  $\epsilon_F^2/N^2$ , which implies a weak correlation of the form

$$\langle \delta \epsilon_j \delta \epsilon_i \rangle = \frac{c}{N} \langle \epsilon_i \rangle^2 \quad (6)$$

among energies. This clearly holds both for women and for men *under the assumption* of the independence of men or women energies, respectively. It can be easily seen that women (men) energies are also weakly correlated, as in eq. (6), if men (women) energies are weakly correlated. We therefore conclude self-consistently that energies are weakly correlated in stable matchings.

In presence of a weak correlation of the form (6), the same procedure, from eq. (3) to eq. (5) leads to  $\epsilon_H \epsilon_F = N(1 + 4c/N)$ . Therefore eq. (5) is exact in the limit  $N \rightarrow \infty$ . We found

numerically that the constant  $c$  is generally negative ( $c \simeq -0.3$  in man optimal states). This correlation is similar to the one occurring among  $N$  variables whose sum is constrained.

Our numerical results indicate that the relation (5) is already satisfied approximatively for a rather small number of pairs. Fig. 1 shows  $\log \epsilon_H$  as a function of  $\log \epsilon_F$  for all stable matchings found in systems of size  $N = 50, 100, 200, 500, 1000$ . The asymptotic result  $\log \epsilon_H + \log \epsilon_F = \log N$  is indicated by lines. The points on the left and on the right of each set of data correspond to the man and the woman optimal GS solutions, respectively.

Fig. 2 shows that the distribution of individual energies in a stable matching agrees very well with the predicted exponential behavior eq. (4).

Although, as shown by the GS algorithm, stable matchings can be very asymmetric regarding the energy of men and women, there is nevertheless a minimum energy of order  $\mathcal{O}(\log N)$  that cannot be reduced further without losing stability. Stable solutions where either men or women possess an average energy of  $\mathcal{O}(1)$  are not possible.

## V. DYNAMICS BETWEEN STABLE STATES

Rotations play a central role in the algorithm which finds all stable states. As for the GS, this algorithm has a man-oriented version and its woman-oriented counterpart. As mentioned a rotation is a cyclic permutation of partners within a subset of persons in a stable matching  $\mathcal{M}$ , which allows to reach a new stable matching  $\mathcal{M}'$ . In the man-oriented algorithm, to which we shall restrict attention, the execution of a rotation raises the energy of any man involved, and lowers the energy of the corresponding woman. In this way, starting from the man-optimal state, the execution of rotations, in all possible orders <sup>2</sup>, allows to reach sequentially all other stable states until the woman optimal one. This process, therefore, runs through the set of stable states shown in Fig. 1 from top (the man-optimal

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<sup>2</sup>A rotation, like the one in eq. (2), can be executed on a stable state only if it is exposed, i.e. only if each man  $\mu_i$  is paired, in that state, with the woman  $\omega_i$  specified in eq. (2), for  $i = 1, \dots, r$ .

state) to bottom (the woman-optimal state).

We can understand this process with a generalization of the GS dynamics. Imagine that in a stable state  $\mathcal{M} = \{(m_i, w_i)\}$ , a woman  $\omega_1$ , for some reason divorces. This event leaves an unpaired man  $\mu_1$ , and the GS dynamics starts again:  $\mu_1$  will run through his list making proposals to the women following  $\omega_1$  until he finds a new woman  $\omega_2$  which prefers him to her partner  $\mu_2$ . This will put man  $\mu_2$  in the same situation as man  $\mu_1$  before. The process will continue, involving other men  $\mu_i$  and women  $\omega_i$ , until a man  $\mu_r$  will make a proposal to woman  $\omega_1$ . Under the GS dynamics, this proposal will be accepted, because  $\omega_1$  is free. It might happen that the new partner of  $\omega_1$  is better than the one she left  $f(\omega_1, \mu_r) < f(\omega_1, \mu_1)$ . In this case the state thus reached will again be stable. The dynamical process described above, exactly represents in this case, the execution of the rotation (2).

On the other hand, if, for woman  $\omega_1$ ,  $\mu_r$  is a worse partner than  $\mu_1$ , the state will be unstable. Indeed  $(\mu_1, \omega_1)$  constitute in this case a blocking pair: both of them would indeed prefer to get together again than to be married with  $\omega_2$  and  $\mu_r$  respectively. We can, in this case, regard the above process as a “virtual” process that does not lead to a new stable state and that leaves the state unchanged.

Note that the probability that the next proposal  $\omega_1$  receives is better than the one she holds in the stable state is  $f(\omega_1, \mu_1)/N \simeq \epsilon_F/N$ .  $\epsilon_F \sim \sqrt{N}$  implies that the probability that any woman improves her situation with a divorce is very small  $O(1/\sqrt{N})$ . With such a small probability divorce is very risky for any woman under the strategies fixed by the GS algorithm (Note, on the other hand, that there will be  $O(\sqrt{N})$  women  $\omega_1$  for which the above process leads to a new stable state).

This dynamics motivates a deeper study of the statistics of rotation lengths. Indeed we can understand the execution of a rotation as the response of the system to the small perturbation which causes the first divorce. While the response is generally linear in equilibrium statistical systems, we shall see that a small perturbation on a stable marriage can cause a large response, i.e. a large change in the system. This is reminiscent of the behavior of self organized critical systems. The statistics of rotation lengths is also interesting to understand

the “geometrical” nature of the organization of stable states. Indeed the rotation length  $r$  measures the “distance” between two stable states, i.e. the number of marriages which differ in the two matchings.

We found that the normalized distribution of the length  $r$  of a rotation satisfies

$$P(r, N) = \frac{1}{r_0(N)} \rho \left[ \frac{r}{r_0(N)} \right] \quad (7)$$

with the typical rotation size scaling with  $N$  as  $r_0(N) \sim \sqrt{N}$ .

The scaled distributions are shown in Fig. 3. All data collapse on a single curve which is remarkably well fit by a Gaussian  $\rho(x) \simeq \frac{\sqrt{\pi}}{2} \exp(-x^2)$  (line). This scaling behavior can be understood considering the above mentioned extension of the GS algorithm. Note indeed that the number of men involved in the process is  $r$ . Using arguments similar to the ones leading to eq. (5), one sees that each man typically needs an additional  $N/\epsilon_F$  proposals. So the total number of proposals received by  $\omega_1$  is  $r/\epsilon_F$ . The length  $r$  of the rotation is obtained imposing that  $\omega_1$  receives  $\sim 1$  proposition. This implies that  $r$  will be typically of the order of  $\epsilon_F \sim \sqrt{N}$ . It also implies that the men’s energy difference between the two states is  $d\epsilon_H \simeq r/\epsilon_F$ , which is of order one. This agrees, apart from logarithmic corrections, with the observation [5] that there are  $\sim N \log N$  stable matchings.

## VI. GLOBALLY OPTIMAL SOLUTION

While there exist powerful numerical algorithms to obtain all stable matchings in a systematic way it is a much harder problem to find the ground state, i.e. the matching with minimal total energy. For an analytical approach it is convenient to introduce the random variable

$$x(m, w) = \frac{1}{N} [h(m, w) + f(w, m)] \quad (8)$$

which is the normalized energy associated with the formation of the pair  $(m, w)$ . Since we are no longer interested in stability considerations it is not necessary to distinguish between

the energy of men and women. In the large  $N$  limit  $x(m, w)$  can be treated as a continuous random variable with distribution  $\rho(x) = \min(x, 2 - x)$  for  $0 < x < 2$ . The problem of finding the minimum of

$$\epsilon(\mathcal{M}) = \sum_{i=1}^N x(m_i, w_i) \quad (9)$$

over all matchings  $\mathcal{M}$  reduces to the bipartite matching problem which has been solved by Mézard and Parisi [7] using the replica technique for the disorder average. Following their approach we obtain

$$\epsilon_{\min} = 1.617\sqrt{N}. \quad (10)$$

On the other hand, minimization of  $\epsilon(\mathcal{M}) = \epsilon_F(\mathcal{M}) + \epsilon_H(\mathcal{M})$  subject to the stability condition, i.e. to eq. (5), yields

$$\epsilon_{\min}^{\text{stable}} = 2\sqrt{N}. \quad (11)$$

Thus giving up the constraint of stability allows for a reduction of energy by 19 percent.

## VII. CONCLUSIONS

We have investigated the classical stable marriage problem which despite its simple definition contains the full complexity of highly frustrated systems like spin glasses. In contrast to the traditional examples of statistical mechanics where the dynamics of the system is governed by a single global quantity, the Hamiltonian, the stable marriage problem fits more naturally into the framework of game theory where the concept of Nash equilibrium plays the central role. The game-theoretical definition of stability leads to the somewhat paradox result that although all individuals do whatever they can in order to maximize their personal benefit the resulting stable states are not the globally best solution. In the large  $N$  limit, we found stable states with total energies ranging from  $\epsilon_{\max}^{\text{stable}} = N/\log N$  to  $\epsilon_{\min}^{\text{stable}} = 2\sqrt{N}$ , whereas the globally best solution has  $\epsilon_{\min} = 1.617\sqrt{N}$ . Within a stable matching the distribution of individual energies, say for the men, is decaying exponentially

where the decay constant is determined by the mean energy of the women, and vice versa. As a consequence, the mean energies of women and men satisfy the simple relation  $\epsilon_H \epsilon_F = N$ . We also studied the distribution of distances between stable matchings in an instance of size  $N$ . This distribution turned out to be a universal function when the distances are scaled with the typical rotation length  $r_0 \sim \sqrt{N}$ . Introducing a simple dynamics, this result also implies that the system is characterized by a non-linear response to perturbation similar to the one observed in self organized critical systems.

There are still many open questions to be investigated in this problem. It would be interesting to compare the distribution of individual energies in the ground state, i.e. the globally best solution, with the one we found in stable matchings. The latter, as shown, are well described by independent variables with a common distribution. On the other hand one expects that, in the ground state, they are much more strongly (anti-) correlated and that individual energies can fluctuate much more wildly (see e.g. [8]).

Another interesting question is how many blocking pairs there are in the ground state since this would be a measure for its degree of instability. A very rough argument suggests that this number is of order  $\epsilon_{\min}^2/4 \sim N$ .

We are also presently investigating a generalization of the model where the assumption that the preference lists of different individuals are uncorrelated is replaced by a more realistic hypothesis.

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## REFERENCES

- [1] D. Fudenberg and J. Tirole, *Game Theory*, The MIT Press, (1991).
- [2] M. Marsili and Y.-C. Zhang, *Fluctuations around Nash Equilibria in Game Theory*, to appear in Physica A (1997).
- [3] D. Gale and L. S. Shapley, *Am. Math. Monthly* **69**, 9 (1962).
- [4] D. E. Knuth, *Mariages Stables*, Les Presses de l'Université de Montréal (Montréal, 1976).
- [5] D. Gusfield and R. W. Irving, *The Stable Marriage Problem: Structure and Algorithms*, The MIT Press (Cambridge, Massachusetts; London, UK; 1989).
- [6] J. Nash, Proc. Nat. Acad. Sci. **36**, 48 (1950).
- [7] M. Mézard and G. Parisi, J. Physique Lett. **46**, L-771 (1985).
- [8] M. Marsili , J. Phys. A: **29** 5405-5419 (1996).

## FIGURES

FIG. 1. Average energy of men ( $\epsilon_H$ ) and women ( $\epsilon_F$ ) for all stable matchings in systems of size  $N = 50, 100, 200, 500, 1000$  (from left to right) on a double logarithmic scale. The analytic result of Eq. (5) is indicated by lines. The  $\diamond$  point below each line corresponds to the Ground State energy.

FIG. 2. Distribution of individual energies of men (open circles) and women (full circles) for all stable matchings in an instance of size  $N = 1000$  on a logarithmic scale. The energies of men (women) are scaled by the their average values:  $x = h(m_i, w_i)/\epsilon_H$  for men, and  $x = f(w_i, m_i)/\epsilon_F$ . The solid line is the analytic result of Eq. (4).

FIG. 3. Scaled distribution of rotation lengths for systems of size  $N = 100, 200, 500, 1000$ . The curve is  $\propto \exp(-x^2/2)$ .







